PSA 1994

PROCEEDINGS OF THE 1994
BIENNIAL MEETING
OF THE
PHILOSOPHY OF SCIENCE
ASSOCIATION

volume one

Symposia and Invited Papers

edited by

DAVID HULL,
MICKY FORBES
&
RICHARD M. BURIAN

1994
Philosophy of Science Association
East Lansing, Michigan

PSA 1994

PROCEEDINGS OF THE 1994
BIENNIAL MEETING
OF THE
PHILOSOPHY OF SCIENCE
ASSOCIATION

volume one

Symposia and Invited Papers

edited by

DAVID HULL,
MICKY FORBES
&
RICHARD M. BURIAN

1994
Philosophy of Science Association
East Lansing, Michigan

In Search of a Pointless Decision Principle[1]

Prasanta S. Bandyopadhayay

University of Rochester

1. Introduction

"Maximizing expected utility (MEU)" is one assumption of (strict) Bayesian decision theory [Savage 1972]. According to the principle of MEU, in a given decision situation, the decision maker should choose one of the alternatives with maximal expected utility [For an excellent discussion of decision theory, see Jeffrey 1990]. However, MEU as the foundation of Bayesian decision theory has been under attack. One counterexample that seems to dispute MEU as a rationality principle is offered by Daniel Ellsberg[Ellsberg 1961]. While discussing what has been dubbed as the Ellsberg paradox, I will consider briefly three decision principles, each of which is different from the MEU principle. These three principles are, (i) Henry Kyburg's Principle, (ii) Peter Gardenfors and Nihls-Eric Sahlin's Maximin criterion (MMEU) [Sahlin 1985, Gardenfors and Sahlin 1988, Sahlin 1993] and finally, (iii) the "weak-dominance principle(WDP)" [Despite Isaac Levi's warning against the term "weak-dominance" of a possible confusion with the similar kind of dominance notion, I stick to this term because of the lack of a better substitute, in private communication] that draws its inspiration from Kyburg's theory which I will endorse [Kyburg 1983a, Kyburg 1983b, Kyburg 1990, Kyburg 1994]. The purpose of this paper is to answer four things: (a) Why are the strict Bayesians who espouse MEU as the standard of rationality really threatened by a paradoxical conclusion? (b) How can an improvement on strict Bayesianism have a better handle on issues like the Ellsberg problem and some other decision problems? (c) How does my decision principle address the Ellsberg problem and some others? (d) What are the ordering properties of the Principle that I advocate and what are its implications?

2. The Ellsberg paradox

Suppose an urn contains 30 red balls and 60 black or yellow balls. The proportion of black balls to the yellow balls in the urn (represented as b) is unknown. One ball is to be drawn from the urn. There are two decision situations, A and B. Each situation has two alternatives. The two alternatives for A are a_1 and a_2. If I choose a_1, I will get $100 if a red ball is drawn and I will receive $0 otherwise. If I choose a_2, then I will get $100 if a black ball is drawn, otherwise I won't receive anything. Under the above circumstances,

PSA 1994, Volume 1, pp. 260-269

I am told that B is the second situation. B, like A, has two alternatives, a_3 and a_4. If I choose a_3, I will get $100 if a red or yellow ball is drawn, otherwise I will receive $0. If I choose a_4, I will get $100 if a black or yellow ball is drawn, otherwise I will receive $0. The two decision situations above can be represented as follows:

	States		
	s_1	s_2	s_3
	1/3	2/3	
Acts	Red	Black	Yellow
A			
a_1	$100	$0	$0
a_2	$0	$100	$0

	States		
	1/3	2/3	
Acts	Red	Black	Yellow
B			
a_3	$100	$0	$100
a_4	$0	$100	$100

MEU does not tell us which course of action one should take in either situation. The expected value of a_1 is the closed interval [100/3, 100/3] and the expected value of a_2 is the closed interval [0, 200/3] provided we represent the uncertainty about the proportion of black balls in the urn which could be anything from 0 to 2/3. In decision situation A, most of the people prefer a_1 to a_2. Likewise, in decision situation B, we cannot say anything about our preference regarding a_3 and a_4 if we stick to MEU. If we represent the uncertainty involved in the proportion of black balls which could be any number from 0 to 2/3, then in B, we get the following intervals. In B, the expected utility calculation yields the closed interval [100/3, 300/3] for choice a_3. We get the closed interval [200/3, 200/3] for a_4 from the expected utility calculation. Again, MEU does not help us. In decision situation B most of the people prefer a_4 to a_3. These two preferences, namely, a_1 over a_2 and a_4 over a_3, are not only intuitive, but also agree, as we have cited, with most people's choices. Why is this decision problem called a paradox?

This decision situation is called a paradox because according to the Bayesian decision theory, one should choose a_4 if and only if one chooses a_2. For any given value of b that represents the uncertainty of the proportion of black balls in the urn, the expected utility of a_2 exceeds that of a_1 if and only if the expected utility of a_4 exceeds that of a_3. This equivalence holds for a Bayesian because he adheres to another Bayesian principle that Leonard Savage calls the "sure thing principle". According to the sure thing principle, the choice between two alternatives must be unaffected by the value of outcomes corresponding to states for which both alternatives have the same payoff. In the matrix above, the outcome a_1 and a_2 are the same for S_3 (i.e., yellow), and the outcome a_3 and a_4 are also the same for S_3. Also, a_1 has the same outcome as a_3, except for S_3 (i.e., yellow); whereas a_2 has the same outcome as a_4 except for S_3. In a situation like this, the sure thing principle requires that a_1 is preferred to a_2 if and only if a_3 is likewise preferred to a_4. Thus, if we take recourse to the standard point valued approach to the MEU principle, then for the Bayesian the rational agent should prefer a_1 to a_2 if and only if the agent should prefer a_3 to a_4, or equivalently

(i) 100 b > 100/3 if and only if (1- b) 100 > 200/3 for all values of b.

Most people however prefer a_1 to a_2 and a_4 to a_3, or equivalently:

(ii) 100/3 >100 b if and only if 200/3 > (1- b)100 for all values of b in the interval [0, 2/3].

There is *no* b that makes (ii) true (except 1/3). On the other hand, if b= 1/3, then the instance of (ii), i.e., 100 > 100 if and only if 200 > 200, is true because both the sides of the equivalence are false. Under this special case when b=1/3, the expected utilities of all of the acts are equal. Then the decision maker should be indifferent between a_1 and a_2 and between a_3 and a_4. However, when b=1/3 one cannot both strictly prefer a_1 to a_2 and a_3 to a4. Thus, the sure thing principle is violated. In what follows, I will discuss three decision principles to see how they address the Ellsberg problem and some others.

3. Kyburg's decision principle

Kyburg provides a decision theory based on probability intervals. According to Kyburg, probability represents a necessary relation between a set of sentences regarded as accepted and a given sentence. For him, this relation does not determine a real number representing the probability of the sentence S as being true relative to knowledge corpus K. Kyburg characterizes the probability of a sentence S relative to a set of sentences K, as the interval [p, q] if and only if the following conditions are met:

(I) S is known in K to be equivalent to a sentence of the form "*a* is an element of set *b*".

(II) "*a* is an element of *c*" is a sentence in K.

(III) The proportion of *c*'s that are *b*'s is known in K in the interval [p, q].

(IV) Relative to K, *a* is a random member of *c* with respect to *b*.

Kyburg provides his central decision rule which he calls the *Principle III*.

The decision maker ought to reject any choice a_i for which there is an act a_j whose minimum expected utility exceeds the maximum expected utility of a_i.

Consider Kyburg's decision rule in the case of the Ellsberg paradox. In A, for example the maximum expected utility of a_1 is 100/3 which is not less than the minimum utility of a_2, which is 0. On the other hand, for B, the maximum expected utility of a_3 is 300/3, which is not less than the minimum expected utility of a_4 which is 200/3. Therefore, for Kyburg, all four acts in the Ellsberg problem are perfectly rational. Though Kyburg's decision rule does not provide a unique decision among these four alternatives, it may often eliminate decisions in other situations. Unfortunately, in the Ellsberg paradox, no action can be eliminated by his central decision rule. Kyburg adds a further decision principle that one should maximize one's minimum gain. By virtue of this principle, a_1 in A and a_4 in B come out to be rational. The additional principle is called the *Minimax* principle. Kyburg thinks that in many cases his decision rule along with the minimax principle, provides good decisions.

4. Gardenfors and Sahlin's decision rule

Gardenfors and Sahlin propose a decision theory that provides a model for an agent's beliefs about the states of world in a decision situation.Their theory, like Kyburg's, is motivated by the thought that the strict Bayesians are too restrictive when they assume a definite probability for each proposition. Unlike Kyburg's theory which is based on probability intervals, the theory assumes that the agent's beliefs can be represented by a set of P of probability distributions, the set of epistemically possible distributions which consist of those measures which are consistent with beliefs the agent has. The set of probability distributions is constrained by a second order measure of epistemic reliability. According to the theory proposed by Gardenfors and Sahlin, the agent exploits those measures of P that are epistemically reliable, i.e. a subset of P/E_1 of P used when making a decision.The theory provides a two-step rule for reaching a decision. First, the expected utility of each choice coupled with each probability distribution P in P/E is calculated and the minimal expected utility of each alternative is determined. Then the largest minimal expected utility among the alternative is chosen. The decision rule Gardenfors and Sahlin provide is the following:

(MMEU) The alternative with the maximum of the minimum expected utility ought to be chosen.

For the Ellsberg paradox, we find that the minimum of the expected utility of a_1 and a_2 are 100/3 and 0 respectively and the minimum of the expected utility of a_3 and a_4 are 100/3 and 200/3 respectively. According to MMEU, we have to choose the largest among the minimum of the expected utilities. a_1 and a_4 become the rational choices according to the MMEU. Interestingly, this matches with the intuition of most people.

5. The third decision rule

I put forward a decision principle within the framework of Kyburg's theory. Like Kyburg's theory, my theory is based on probability intervals. The probability of a proposition S being true as an element of our knowledge corpus K belongs to the interval [p, q]. If a rational agent has complete knowledge about an event, its associated probability interval would be reduced to a point, whereas if his information is incomplete, the associated probability interval would be wider. If a rational agent is completely ignorant of the event, then his complete ignorance can be represented by the interval from 0 to 1.

Since I am proposing a decision principle it is worthwhile to provide a connection between the degrees of belief and utilities with actions. I assume that the utility function U is a real-valued function defined over sentences. Then the outcomes of decisions can be regarded as certain propositions of coming true. In my theory, as in Kyburg's, since probabilities are intervals, expected utilities have to be construed as intervals. The expected utility of a sentence S is the interval consisting of the utility of S's being true multiplied by the lower probability of S, and the utility of S's being true multiplied by the upper probability of S. If the probability of S is the closed interval [p, q], and utility for S is U, the expected utility of S is [Up, Uq].

Since the expected utilities are intervals, it does not make sense to maximize intervals in the same way which we maximize expected utilities for points. Although we cannot maximize an interval in the usual way, Kyburg suggests a principle that seems to be rational in this context. That is, one ought not to choose an action whose maximum expected utility is less than the minimum expected utility of some other action. In decision situations, Kyburg's principle eliminates many acts as irrational. I want to strengthen Kyburg's theory by adding further conditions to his theory, and including Kyburg's central principle. I call my decision principle the *Weak dominance principle*:

> *For all closed intervals I and J, if min(I) < min (J) and max(I) < max(J), and either min(I) < min(J) or max(I) < max(J), then one must choose the act corresponding to J.*

The strategy of the paper is to argue that neither do I agree with strict Bayesianism nor with Gardenfors and Sahlin type theory nor with Kyburg's theory in all its nuances. (I) Recall the Ellsberg problem. In this problem, the intuitive decision is to prefer a_1 to a_2 in decision situation A and a_4 to a_3 in decision situation B. In A, the expected utility a_1 is [100/3, 100/3] whereas, the expected utility of a_2 is [0, 200/3]. In B, expected utility of a_3 is [100/3,300/3] and expected utility of a_4 is [200/3, 200/3]. In the Ellsberg paradox, Kyburg's Principle and WDP cannot eliminate any of the actions as irrational. Kyburg suggests that if one uses the *Minimax principle*, then he can choose a_1 in A and a_4 in B. This decision agrees with the intuitions of most people. I will argue that although the *Minimax* principle yields a unique and intuitive solution in the Ellsberg paradox, the *Minimax* principle cannot provide a general and intuitive solution to all decision situations.

Consider an Ellsberg type decision situation with the same probability distributions for red, black and yellow balls as in the Ellsberg paradox. The only difference between the former and the latter is the utility matrix

	States		
Acts	1/3	2/3	
	Red	black	Yellow
a_1	1	0	0
a_2	0	300	0

Based on probability intervals, the expected utility of a_1 is [1/3, 1/3] and that of a_2 is [0, 600/3]. According to the *Minimax* principle, a_1 is the correct decision, although the majority of the people will recommend a_2 as the correct choice. In the Ellsberg paradox, our intuition matches with the choice demanded by the *Minimax* principle, whereas in this Ellsberg type situation, most people would not agree with the decision provided by the latter principle. Our intuition varies from one case to the next. Under these two situations, our decisions are made based on the different psychological factors affecting our relevant decisions. In the original Ellsberg paradox, despite the uncertainty about the proportion of black balls to yellow balls (it can be anything between zero to two thirds), my analysis of the situation indicates that most of the people prefer a_1 to a_2 since the expected utility of a_1 lies at the center of the closed interval of a_2. On the other hand, in the Ellsberg type situation just presented, the expected utility of a_1 is the closed interval that lies more close to zero of the closed interval of a_2 than to 600/3 of the same interval. So in the latter situation, a_2 seems to be the rational decision for most people.

Even in the Ellsberg type situation, there may be some people who choose a_1 to be the rational decision. A hungry person may opt for a_1 in the Ellsberg type situation. For him, buying a hamburger for that price rather than choosing a_2 for a possible larger fortune seems to be rational. However, those who are relatively well off and whose

main concern is not buying hamburgers and other cheap things may opt for a_2. For them, gambling for a large fortune is much more attractive, hence rational. According to the *Minimax* principle, any agent ought to choose a_1 in the Ellsberg type decision situation. On my account, whether the agent will choose a_1 or a_2 depends on the psychological factors affecting his decision. If the agent is risk averse, he may choose a_1. On the other hand, if the agent is risk prone, he may opt for a_2. So, in the Ellsberg type decision situation, though the *Minimax* principle demands a_1 to be the correct decision, this may not be the right decision in a given decision situation for a particular agent. So, one should not apply the *Minimax* principle expecting to get both a unique and intuitive decision in all decision situations.

(II) I will provide a utility matrix associated with two sets of probability distributions(e.g., p_1 and p_2) to see how my decision criterion fairs amidst the two other decision rules. For the sake of the discussion, I denote the utility matrix with two probability distributions by M.

	States	
Acts	S_1	S_2
a_1	1	-1
a_2	-1	1
a_3	0	0

Probability Distributions

$p_1(s_1)=.4$
$p_1(s_2)=.6$
$p_2(s_1)=p_2(s_2)=.5$

Expected utility(a_1)= [-.2, 0]
Expected utility(a_2)=[0,.2]
Expected utility(a_3)=[0,0]

According to Kyburg, any of the choices are perfectly rational since we have no way of determining that any of the options are irrational based on his principle[For Levi too, all options are both E-admissible and S-admissible. Therefore, all options are feasible. For lack of space, I have to leave out Levi's well-articulated decision theory from the paper. For Levi's theory, see Leŵi 1974, Levi 1988]. For Gardenfors and Sahlin a_2 and a_3 are the only feasible options. The MMEU criterion rejects a_1 as the rational choice because it is not the largest of the minimum of expected values. My decision rule, by contrast, demands a_2 to be chosen as rational because that option is no worse than its alternatives under any circumstances and it is at least better than any alternative under some circumstances. So, a_2 seems to be the natural choice among all other alternatives and my decision principle alone agrees with the choice.

6. Ordering relationship of the decision rule

Recall WDP: For all closed intervals I and J, if min(I) < min (J) and max(I) < max(J), and either min(I) < min(J) or max(I) < max(J), then one must choose the act corresponding to J. The principle induces a strict partial ordering, i.e. it is irreflexive and transitive on the intervals to which it is applied. The principle induces irreflexive ordering on intervals since the interval I cannot be better than itself. Therefore, that the ordering relation is irreflexive follows from the definition of <. The principle produces an ordering on intervals which is also transitive. Consider the first conjunct of the antecedent. That conjunct consists of two conjuncts, namely min(I) < min(J) and max(I) < max(J). For all I, J, and K, if they have a transitive relation, then if I< J and J< K, then by transitivity, I necessarily < K. Suppose I< J, then min(I) < min(J). Since J < K, min(J) < min(K).Therefore, min(I)< min(K). By the similar argument, max(I) < max(J). Consider the other limb of the conjunction, min(I) < min(J) or max(I) < max(J).

Take its first disjunct, i.e. min(I) < min(J). For all I, J and K, if min(I) < min(J), as it has been shown that min(J) < min(K), hence, the claim min(I) < min(K) is proved. Take the rest of the disjunct, i.e. max(I) < max (J). For all I, J, and K, if max(I) < max(J) and max(J) < max (K), then show that max (I) < max(K).We have shown before that max(J) < max(K). Therefore, max(I) < max(K). Q.E.D. One interesting ordering relation the weak-dominance principle produces on intervals is that it forms a lattice. A strict partially ordered set is called a *lattice* if and only if for any two elements a and b, there is a least upper bound and a greatest lowest bound for a and b.

In the given framework we can characterize strict preference and indifference relations between two acts. We rewrite WDP and divide it into two parts.

(i) Given that the two intervals are not disjoint, if their intervals have the same maximum, choose the act corresponding to the interval with higher minimum.

(ii) Given that the two intervals are not disjoint, if their intervals have the same minimum, choose the act corresponding to the interval with higher maximum.

The two conditions (i and ii) give *strict preference orderings* over acts. The agent is *indifferent* between two acts iff both max(I) = max(J) and min(I) = min(J). In other words, selection of one act over the other, in this situation, is irrelevant to our principle.There are many acts for which neither the agent has strict preference nor is he indifferent between them. We will return to these acts which are *incomparable* to one another.

When an agent is indifferent between two acts, one of the proposed conditions to eliminate all acts except one is to consider whether the choice the rule recommends remains invariant even after the application of the condition called the *mixture condition*. The intuition behind this condition is that if the agent is indifferent between two acts, then the agent will be indifferent between them and the third act of tossing a fair coin and doing the first act on heads and the second on tails. Suppose the utilities of both a_1 and a_2 are [1,0]. Then, if we add the "mixed" act of flipping a fair coin and doing a_1 on heads and a_2 on tails, we get the following matrix.

	s_1	s_2
a_1	0	1
a_2	1	0
a_3	1/2	1/2

The values for a_3 are calculated by considering that its utility under either state is the expected utility of a bet on a fair coin that pays 0 and 1.

We are here no longer indifferent among three acts after the application of mixture condition, though pairwise, we may be indifferent to one another. For an agent, a_1 and a_2 are still indifferent, but neither a_1 is indifferent to a_3 nor is a_2 to a_3. Among three acts, a_3 has the higher minimum, i.e. 1/2, whereas both a_1 and a_2 have higher maximum than a_3. Our rule does not recommend any action being rational in this situation. We call these acts, a_1, a_2 and a_3, incomparable leaving open the possibility that the act which an agent will choose depends upon his or her mental make up. In fact, in the spectrum between two extreme acts(same minimum, but different maximum, and same maximum, but different minimum), most acts fall into the category of being incomparable. Suppose a_1 is [1, 10,000] and a_2 is [100, 120]. Assuming that a_1

and a_2 represent expected utilities of both acts, the *optimistic reasoner* may choose a_1 arguing that since the true value must lie in between the closed interval, why not take the risk? If we hit the Jack-pot, then we may get 10,000. So, the optimistic reasoner chooses a_1. The *pessimistic reasoner*, on the other hand, may opt for a_2. He may contend that if we choose a_1 and unfortunately a_1 is not the true state of nature, then he may loose a huge amount of money. Then, why not go for a_2 which lies in between the interval [100, 120]? Our principle does not provide us a unique solution to these cases. This does not represent any drawback on the part of the principle rather it shows that our principle is realistic in capturing the true state of nature.

7. Relationship among the three decision rules

I will consider whether any decision principle is entailed by any other principle by providing an intuitive justification for the connection among different principles. I will frequently refer to Sahlin's utility matrix with two probability distributions as M* which is as follows:

	States	
Acts	s_1	s_2
a_1	12	-10
a_2	-9	11
a_3	0	0

Probability distributions

$p_1(s_1) = .04$, $p_1(s_2) = .06$;
$p_2(s_1) = .06$, $p_2(s_2) = .04$;

Expected utility (a_1)= [-1.2, 3.2]
Expected utility (a_2)= [-1,3]
Expected utility (a_3) = [0,0]

For Kyburg's principle, all acts are rational [Levi, however, thinks a_1 and a_2 to be E-admissible and only a_2 to be S-admissible]. According to the MMEU, a_3 is the correct choice. For the sake of the discussion, I will reformulate some of the above decision principles in the following manner:

(1) If two intervals are disjoint, reject the one with lower maximum.

(2) Given that the two intervals, namely I and J are not disjoint, then if max(J) > max(I), but it is not the case that min(J) is < min (I), then choose the act corresponding to J.

(3) Choose the one with highest minimum.

Principle 1 is equivalent to Kyburg's principle. Principle 2 represents WDP. Gardenfors and Sahlin accept principle 3, that is, the *Minimax* principle. Let's begin with rule 1. Rule 1 implies neither rule 2 nor rule 3. Principle 1 is a special case of the principle 2[needless to rehearse the obvious steps of the proof]. For example, where I=[2,3] and J = [4,4], both rules reject I as a rational act. Consider now J' = [3,4] in addition to I. Rule 1 rejects neither of them, whereas rule 2 rejects I as a rational act. Principle 1 does not imply principle 3. If we consider M*, then we find that all acts are rational from the standpoint of Kyburg's principle (i.e., principle 1). But, neither act a_1 nor act a_2 is recommended by principle 3.

Consider rule 2. Rule 2 implies rule 1, for the reason given above. Clearly, rule 2 does not imply rule 3. Consider M. In M, rule 2 recommends a_2 as the rational choice whereas rule 3 accepts both a_2 and a_3 as rational choices. Consider rule 3. Rule 3 does not entail rule 1. Consider M*. According to rule 3, a_3 is the correct choice. It shows a clear violation of rule 1 since all acts are rational according to rule 1.Nor does principle 3 entail principle 2 either. Suppose c_1 and c_2 are two acts with the closed intervals, [.2, .5] and [.2, .6]. According to principle 2, we ought to choose c_2, whereas for principle 3, we cannot reject either of them. In short, principle 2 entails 1 whereas for the rest of them the entailment relationship is not that straightforward.

8. Concluding remarks

Based on the weak-dominance principle, which rests on the interval notion of probability, I argued for both: (i) why should not one expect to reach a unique option from two different sets of options in connection with the Ellsberg paradox and (ii) that, at least under one circumstance, the principle gives better results than the rest of the principles discussed above. I showed that the principle is strict partially ordered and also discussed that it forms a lattice.

Note

[1]The author acknowledges his debt to Prasun Basu, John G. Bennett, Abhijit Dasgupta, Deepnarayan Gupta, Jack Hall, Henry Kyburg Jr., Isaac Levi, Patrick Maher, Michael Mathias, Abhaya Nayak, Nihls-Eric Sahlin, Paul Weirich for their suggestions and comments on earlier versions of the paper and to Peter G. Found for suggesting to him the title of the paper. John G. Bennett deserves special mention for helping him in thinking through the difficult areas of decision theory.

References

Ellsberg, D. (1961), "Risk, Ambiguity, and the Savage Axioms" *Quarterly Journal of Economics*, 75, 528-557. Also, in Gardenfors and Sahlin (eds).

Gardenfors, P.. and Sahlin, N. (1988), "Unreliable Probabilities, Risk Taking, and Decision Making" in Gardenfors and Sahlin (eds). *Decision, Probability, and Utility*, Cambridge: Cambridge University Press.

Kyburg, H.E., Jr. (1983a), *Epistemology and Inference*, Minneappolis: University of Minnesota Press.

---------. (1983b). "Rational Belief" in *The Behavioral and Brain Sciences*, Vol 6: No 2, 231-273.

---------. (1990). *Science and Reason*, Oxford: Oxford University Press.

---------. (1994)."Believing on the Basis of the Evidence" in *Computational Intelligence*, Vol 10; No 1.

Levi, I. (1974), "On Indeterminate Probabilities", in *Journal of Philosophy* 71: 391-418.

---. (1988), *Hard Choices*, Cambridge: Cambridge University Press.

Jeffrey, R. (1990), *The Logic of Decision*, Chicago: University of Chicago Press.

Sahlin, N. (1985), "Three decision rules for generalized probability representations" in *The Behavioral and Brain Sciences*, Vol 8, No 4: 751-753.

-----. (1993), "On Higher Order Beliefs" in *Philosophy of Probability* (ed) J.P. Dubucs, Holland: Kluwer Academy.

Savage, L. (1972). *The Foundations of Statistics*, New York: Dover Publications, Inc.