Kant und die Berliner Aufklärung

Akten des IX. Internationalen Kant-Kongresses

Band II: Sektionen I–V

Herausgegeben
im Auftrag der Kant-Gesellschaft e.V.

von

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Walter de Gruyter · Berlin · New York

2001

Incongruent Counterparts and the Nature of Space: Demystifying their Reappearance in Kant's Writings

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Overview

Two objects are called incongruent counterparts if and only if one object is a mirror image of the other, and yet one could not occupy the region of space just vacated by the other. (For a sophisticated discussion on this and other related notions of incongruent counterparts, see Earman 1991) In his pre-critical writings, especially in the "Regions in Space" (1768), Kant uses the existence of incongruent counterparts to argue that space is absolute, i. e., not relational, and intuitive, i. e., not a concept.[1] By "absolute space", Kant means space that exists independent of matter.[2] In the critical period of his career, incongruent-counterparts reappear in the *Prolegomena* (1783, § 13). Commentators like Beck[3] contend that Kant employs incongruent counterparts in 1783 to reinforce the thesis that space is both absolute and intuitive.[4] Here, Kant takes "absolute space"[5] to be one which exists over and above a

[1] In the *Dissertation* (1770), Kant also discussed incongruent counterparts. However, I prefer Kant's 1768 article as a representative article on this topic.

[2] "Regions in Space". It is controversial whether Kant's notion of space as absolute is the same as that of Newton. See Friedman on this point. M. Friedman, *Kant and the Exact Sciences*, Mass., Harvard University Press, 1992

[3] For Beck's view, see Beck, *Early German Philosophy: Kant and his predecessors*, Cambridge, Mass., Harvard University Press, 1969. I thank Beck for giving me the opportunity to speak with him several times regarding this point till his death in 1997. Broad seems to be another commentator subscribing to this interpretation. He takes Kant to be arguing in the *Prolegomena*, first, that incongruent counterparts show that space is absolute and, second, that absolute space has a property that is inconsistent with it being a thing in itself. C. Broad, *Kant: An Introduction*, Cambridge, Cambridge University Press, 1978.

[4] The focus of the paper is to address whether for Kant space is absolute. I assume that there is no controversy among Kantian commentators regarding the question of intuitivity of space. This assumption will also help simplify the discussion.

[5] *Prolegomena*. See Earman's article, "On the Other Hand: A Reconsideration of Kant, Incongruent Counterparts, and Absolute Space" (1991) for a similar sense of absolute space. This article occurs in J, Van Cleve and R. Frederick (Eds.) *The Philosophy of Right and Left*, Kluwer, Academic Publishers, Dordrecht, 1991.

[6] There might be differences in the senses of "absolute space" as Kant used it in 1768 and in 1783. For example, Meerbote thinks that whatever may be the meaning of absolute space, Kant's sense of absolute space in 1783 is not definitely the Newtonian sense of absolute space (in a private conversation). For the sake of discussion, these subtle differences, if any, between the different senses of absolute space won't affect the main line of the argument broached in the paper. I think that for Kant space as being ontologically prior to a system of relations among objects is the underlying theme of both his writings (1768 and 1783).

system of relations among objects.[6] That is to say, space is not reducible to a system of relations.

Two questions then arise. First, why does Kant recycle incongruent counterparts to argue the same point that space is absolute and intuitive and second, why does Kant not use them in the *First Critique* (1781), to argue the same point about space?[7] Commentators like Meerbote[8] disagree with the significance of these questions. Meerbote contends that incongruent-counterparts play different roles in the "Regions in Space" and the *Prolegomena*. In the "Regions in Space", Kant argues that space is both absolute and intuitive, whereas, it is alleged, in the *Prolegomena*, space is both *relational* and intuitive. I call this latter account the relational account of space. I contend that the relational account fails to provide a *unified* approach to Kant's philosophy of geometry since it is incompatible with Kant's *anti-reductionist*[9] research program in philosophy of mathematics.[10]

To respond to the first question, "Why does Kant recycle incongruent counterparts in his writings?", I argue that Kant's notion of proof underwent a change over a period of fifteen years (1768-1783). I think that a satisfactory answer to this question will also provide a satisfactory answer to the second question: Why does Kant not use them in the *First Critique* (1781), to argue the same point about space? First, I interpret his 1768 argument for absolute space as a *reductio* of the relational theory. Second, I take his 1783 argument for space as a *direct proof* for space being absolute. I attribute the change in his style of proof to the "transcendental philosophy", which is well developed at the time of the *First Critique* (1781). This change prompts him to use incongruent counterparts for the first time in the critical period of his career. My account reinstates Beck's interpretation about the absolute character of space, thus providing both a historically interesting and a philosophically credible account of Kant's motivation for recycling incongruent counterparts.

A Reconstruction of Beck's Interpretations

"Regions in Space"

Kant asks us to consider a thought-experiment. Suppose God created only one solitary hand and nothing else in the universe. If one is a relationist, then one will argue that there exists no other object with respect to which one can tell precisely that it is a left or a right hand. Hence, on the relationist reading, the exact nature of the hand is indeterminate. Kant contends, however, that al-

[7] All references including subsequent quotations of the *Critique of Pure Reason* are from the translation by P. Guyer and A. Wood, Cambridge University Press, 1998.

[8] Meerbote, "Kant on Intuitivity" in *Synthese*, 47 (1981), pp. 203-228.

[9] This catchy expression, "anti-reductionism" is due to Brittan. See Brittan, *Theory of Science*, Princeton University Press, 1979. However, I don't know whether Brittan will agree with my reading of Kant's anti-reductionism here.

[10] There is a further reason for arguing that Kant cannot be a relationalist during the period of the *Prolegomena*. See footnote 19.

though there is only a solitary hand, it must be either left or right. If the hand, he rejoins, is necessarily either left or right, then that hand must be left or right with respect to a thing. Since in the universe there is nothing except the solitary hand, the hand must be left or right with respect to a space that exists over and above the solitary hand.[11] Hence, it follows that there is absolute space over and above the existing objects that are embedded in that space.

Prolegomena

Kant dramatizes the puzzle of incongruent counterparts with a pair of gloves. One is a left glove and the other one is a right glove. Both look alike and have the same relation to their corresponding parts. For example, the angle between the thumb and the index finger is one and the same for both the gloves, the length of each index finger is also the same and so on. However, one cannot be replaced by the other, since they can't occupy the same space. What Kant intends to say is that there is no rigid motion of the left glove that can superimpose it into the right glove while the left glove shares the same surface with the latter. Although from the perspective of relations between the two gloves, they are equal and the same, we can differentiate one from the other. How is this possible? Kant writes, "Here, then, is an internal difference between the two [gloves], which difference our understanding cannot describe as internal and which only manifests itself by external relations in space".[12]

The question may arise why does the external relation to space show that space is absolute rather relational? The reason for Kant's considering space as absolute is that the gloves' external relation to space is in fact an implication of their relation to the whole space. Kant writes, "Space is the form of the external intuition of this sensibility, and the internal determination of every space is possible only by the determination of its external relation to the whole of space of which it is a part ..."[13]

Here, by "the whole of space", Kant means that it is the nature of *space as a whole* that determines the handedness of the gloves rather than the existing relation of the gloves to other objects that determines their handedness.[14] Only philosophers who subscribe to the absolute theory of space can accept the idea of space as a whole determining the direction of a hand. So

[11] A more careful formulation of this point can be found in R. Frederick's article, "Introduction to the Argument of 1768". Here, he writes, "[h]ence, the hand must be left or right at least *partly* in virtue of its relation to absolute space". Emphasis is mine. This passage occurs in J Van Cleve and R. Frederick (Eds.) *The Philosophy of Right and Left*, Kluwer, Academic Publishers, Dordrecht, 1991.

[12] *Prolegomena § 13*. This passage occurs in J, Van Cleve and R. Frederick (Eds.) *The Philosophy of Right and Left*, Kluwer, Academic Publishers, Dordrecht, 1991.

[13] *Prolegomena § 13* (see footnote 12)

[14] Here, I am influenced by Nerlich's interpretation. See G. Nerlich, *The Shape of Space*, second edition, Cambridge University Press, 1994.

I take Kant to be saying that space is absolute, not a system of relations among objects.[15] This is also Beck's reading of Kant's argument. However, this raises the question, "why does Kant recycle incongruent counterparts at least twice to argue that space is both absolute and intuitive"? Interestingly, the relationist account provides an answer, however wrong, to the question.

The Relationist Contention

Commentators like Meerbote disagree with the above interpretation of Kant's 1783 argument. Referring to the last but one passage I have just quoted from the *Prolegomena*, he contends that Kant explains the inner difference between incongruent counterparts in terms of a difference determinable by means of non-discursive external spatial relations. And this spatial difference, according to Meerbote, can be expressed relationally without invoking absolute space. He concludes "that Kant's resolution consists of saying that there is *no non-relational* difference in the case of incongruent-counterparts, and that the difference there must be *relational* and intuitive".[16] Contrasting Kant's position discussed in the 1768 paper with this change, Meerbote says "[on] this reading, Kant, who earlier in his thinking on these matters had argued that left-handedness, for example, must be understood as a non-relational spatial property, is now agreeing with Leibniz that it is relationally determinable".[17] So, Meerbote concludes that Kant's 1783 argument is evidence for Kant's agreement with Leibniz regarding space being relational.

What is Wrong with the Relationist Account?

I argue that Meerbote's account overlooks a deep philosophical difference between the two research programs, Kant's and Leibniz's. Kant's research program in philosophy of mathematics is entirely anti-reductionist, whereas Leibniz's research program is thoroughly reductionist. According to Kant, all mathematical propositions are synthetic (*Critique*, B 14). Kant admits that although in a mathematical deduction, one step follows from another analytically, that is, in accordance to the law of contradiction, the premises to begin with and the conclusion we end up with in the deduction are synthetic. Leibniz contends, to the contrary, that all mathematical propositions are

[15] J. Buroker accepts this interpretation. However, she has gone further to contend that this provides a justification for Kant's conclusion that space must be merely ideal. See, Buroker's article, "The Role of Incongruent Counterparts in Kant's Transcendental Idealism" in J Van Cleve and R. Frederick (Eds.) *The Philosophy of Right and Left*, Kluwer, Academic Publishers, Dordrecht, 1991. Van Cleve, by contrast, disagrees both with her conclusion and some consequences of her view. See Van Cleve's article "Incongruent Counterparts and Things in Themselves" in J Van Cleve and R. Frederick (Eds.) *The Philosophy of Right and Left*, Kluwer, Academic Publishers, Dordrecht, 1991.

[16] *Synthese*, 47 (1981), pp. 203-228. Emphasis is mine.

[17] Ibid.

analytic. On Leibniz's view, all mathematical propositions can be reduced to definitions and the principle of contradiction.[18] By "mathematical propositions", both Leibniz and Kant include algebraic, geometric and arithmetical propositions.

I argue that when Kant holds that mathematical propositions without any exception are synthetic he has in mind how objects referred to in a theory or in a sentence are capable of being intuited in an objective fashion in a spatio-temporal framework. Kant thinks that mathematical propositions involve synthesis, which, in turn, involves intuition. What he means by intuition is that the object of our discourse or objects referred to in a theory are capable of being experienced by human beings. Objects of discourse must be objects that are *knowable*. However, according to him, objects that are knowable are those that are capable of occupying determinate space-time positions. Objects that are capable of occupying determinate spatio-temporal position are objects of our sensible intuition. He distinguishes this type of object from the other kinds like "God" and "soul" which are not subject to our possible sensible intuition.

Sensible objects have objective reality that presuppose space and time as *a priori* conditions of experience. Unless space and time exist over and above the system of relations among objects, they cannot be *a priori* conditions of our experience. On Kant's account, systems of relations are not given *a priori* since they depend on objects and their existing relations. If space and time were a system of relations among objects, they would be bound to change due to corresponding changes in the system of relations, thus failing to be *a priori* conditions of our experience.

Kant contends that the real difference between incongruent counterparts is manifested to us because it is with regard to the absolute "objective" space that we are able to distinguish the left glove from the right glove. According to him, it is the nature of space that contributes to the intrinsic difference between incongruent counterparts. This is what Kant means when he says that the determination of the difference between incongruent-counterparts can be understood only with respect to their external relations to the "whole of space". Here, the whole of space is not something that is reducible to its parts or to a system of relations among objects. Rather its parts or the existing system of relations is possible because of the existence of the whole of space. In short, for Kant, space is not something that is reducible to a system of relations among objects, whereas, for Leibniz, space is nothing other than a system of relations among objects. Hence, while Kant is an anti-reductionist in mathematics, Leibniz is a reductionist.

In this section, I have argued that the relationalist reading of the *Prolegomena* is mistaken.[19] If the relational account is wrong, then it leads us back to

[18] See the second letter to Clarke, *The Leibniz-Clarke Correspondence*, ed. H. Alexander, Manchester University press, Barnes and Noble, New York, 1965.

[19] Further, we can account for the difference between Kant and Leibniz with regard to the respective questions they address. Kant is primarily interested in the questions; "... what are

the question I have raised before, viz, why does Kant recycle incongruent-counterparts to argue the same point that space is both absolute and intuitive in at least two occasions without being repetitious? An answer to this question lies in Kant's changing views on the notions of proof.

Changes in Kant's Notions of Proof

Recall Kant's thought-experiment of the existence of a solitary hand. Kant contends that if the relational theory were true, then we had to conclude that the hand in question was indeterminate. We have seen before that Kant does not think that the hand is indeterminate since if it were the only one hand then

space and time? Are they actual entities? Are they only determinations or relations of things ...? (*Critique* B 38)". Kant thinks that space is ontologically prior to its objects and is not reducible to a system of relation among objects, possible or actual. I think that Kant's interest in the metaphysical investigation about the nature of space leads to his epistemological investigation. So, it will be a mistake to identify Kant as a philosopher whose only query with respect to space is metaphysical. If we continue to follow what he has said in the *Critque* B 38, then we will see that clearly: "Are they [i. e., space and time] only determinations or relations of things, yet ones that would pertain to them even if they were not intuited?" In *The Metaphysical Exposition of Space*, Kant raises an epistemological point regarding our ability to represent an empty space (i. e., a space without any object in it) and at the same time our inability to represent an object that does not occupy a space. Although, Kant raises epistemological questions regarding the nature of space, he raises them as a follow-up of his metaphysical questions. This is why, I call Kant's interest primarily quasi-metaphysical. Leibniz, on the other hand, is primarily keen on the epistemological issue of spatial and temporal relations among objects. He provides an answer to the question, "how is our knowledge of space possible?" In his Fifth letter to Clarke, Leibniz writes, "I will here show, how men come to form themselves the notion of space. They consider that many things exist at once and they observe in them a certain order of co-existence, according to which the relation of one thing to another is more or less simple. This order, is their *situation* or distance. When it happens that one of those co-existent things changes its relation to a multitude of others, which do not change their relation among themselves; and that another thing, newly come, acquires the same relation to the others, as the former had; we then say, it is come into the place of the former; and this change, we call a motion in that body, wherein is the immediate cause of the change. ... And supposing or feigning, that among those co-existents, there is a sufficient number of them, which have undergone no change; then we may say, that those which have such a relation to those fixed existents, as others had to them before, have now the *same place* which others had. And that which comprehends all those places is called *space*." Leibniz argues and I think rightly so that we cannot have knowledge of space without knowing the existence of some object, possible or actual, occupying the space. Recall Kant's thought-experiment about the solitary hand and see how Leibniz would have responded to it. On Leibniz's reading, one cannot determine or know whether the hand is right or left since it is not related to an object with respect to which we can determine whether it is a left or a right hand. Hence, on this reading, the solitary hand is indeterminate. Since according to Leibniz, we cannot *determine* or *know* whether the hand is left or right, I call Leibniz's interest in matters related to the notion of space as primarily epistemological. However, Leibniz is also interested in the ontological question regarding space. I argue that his ontological investigation is an off-shot of his epistemological investigation. Since, according to the relational account, we cannot know whether the solitary hand is left or right, we can make a plausible metaphysical claim based on this epistemological argument that space cannot be ontologically prior to objects. If there is something called a space, then, on the relationalist reading, it must be a "fiction" built from a system of relations among objects. So with regard to space, Leibniz's interest is quasi-epistemological, while Kant's interest is quasi-metaphysical.

it must be either left or right. Hence, it follows that space must be absolute over and above the solitary hand because with respect to space alone the solitary hand can be completely determined to be either left or right. Thus, Kant's thought-experiment intends to show the absurdity of the relational theory.

I interpret this argument to be an *indirect* or *reductio* argument against the relational account.[20] Kant begins the thought-experiment with the assumption that space is relational. Call this assumption premise one (P1 for short). If space were relational, then we would have failed to determine the handedness of the solitary hand. P2 says if the relational account were true, then we should have declared that the hand in question is indeterminate. We derive P3 from P1 and P2 that the hand in question is indeterminate. The solitary hand, as we all know, so argues Kant, is *not* indeterminate. Indeed, P4 is the assertion that it must be either left or right. In other words, P4 says that it is logically impossible for a hand to be indeterminate. P3 and P4 contradict one another. Hence, according to Kant, relationalism must be false. One could easily find fault with Kant's argument. However, my purpose is only to reconstruct its form.

I contend that Kant's notions of proof undergo a change over a period of fifteen years (1768-1783). This change prompts him to use incongruent counterparts in at least two different ways during his career. When he writes "Regions in Space," to defend the absolute theory he provides a reductio of the relational theory. When he, by contrast, writes the relevant section of the *Prolegomena*, he furnishes what Beck calls (in private discussion) "a crucial test"[21] for the absolute theory. I interpret Beck's claim that in the *Prolegomena* Kant provides a crucial test to mean a "direct proof" for absolute space. A direct proof is better understood if it is contrasted with an indirect proof. An indirect proof is one where we set up the situation so that the opposite of what we want to prove is assumed. Then we prove that this contradicts some other basic feature of the problem. Therefore, since both cannot be true, the assumption we made must be false. Thus its opposite must be true. In a direct proof, we don't do that. What is a direct proof?

Two Senses of a Direct Proof

There are two distinct senses of a direct proof in mathematics. The first is called an *existence proof* and the second is called a *constructive proof*.[22] An existence proof establishes the existence of *some* entity without informing us

[20] W. Harper also takes this argument to be a reductio of the relational theory. See his article "Kant on Incongruent Counterparts" in J Van Cleve and R. Frederrick (Eds.) *The Philosophy of Right and Left*, Kluwer, Academic Publishers, Dordrecht, 1991.

[21] Beck told me in a private conversation that Kant's experiment with incongruent counterparts is an analogue of Newton's bucket-experiment to establish that space is absolute. In this sense, the existence of incongruent-counterparts is a crucial test for Kant.

[22] There is a vivid discussion of this distinction in B. Bunch's *Mathematical Fallacies and Paradoxes*, Van Nostrand Reinhold Company, New York, 1982. This distinction lies at the heart of intuitionist mathematics. For a sophisticated discussion on intuitionism, see M.

of how to find it. Here is an example of an existence proof.[23] There is a well-known problem in elementary mathematics whether we can prove that in any sufficiently large city at least two people must have exactly the same number of hairs on their heads. In the case of a big city like New York City all one need to know is that the number of hairs on any given head is less than the city's population of roughly 10, 000000. If each person is tagged by his or her specific number of hairs, at least two people must be tagged by the same number. That is, two people must have the identical number of hairs.

One proof of this problem adopts the form of an existence proof. Consider, for example, the residents of New York City. Put persons with no hairs on their heads in group zero, one hair on their heads in group one, those with two hairs in group two, and so on. By hypothesis, we will need at most ten million groups to accommodate everyone. Start with the person with the fewest hairs on his or her head. Check to see if there is another person with the same number. If so, our proof is complete. If not, move on to the person with the next fewest hairs on his or her head and repeat the process. We have more people than hairs. So we will have to find a match before we get to the last person of the population. QED.

Examples of existence proof can be also found in geometry. Suppose I want to prove that if a triangle is not isosceles, then the bisector of one angle must meet the perpendicular bisector of the opposite side in a point *outside* the triangle. The proof shows that if the triangles are not isosceles then there is a point that lies outside the triangle. The proof does not pinpoint the exact location of that point outside the triangle. On my proof, I have to show only that such a point exists outside the triangle. This kind of proof is known as an existence proof. Sometimes mathematicians are not satisfied with this existence proof. They offer a constructive proof that will show, if possible, where exactly, that *particular point* lies outside the triangle. Recall the problem where we are confronted to prove that there are at least two people with the same number of hairs on their heads. In this scenario, mathematicians interested in a constructive proof want to know, if possible, *exactly* which two people have the same number of hairs. So there are two senses of a direct proof, an existence proof and a constructive proof.

What Does Kant mean by a Direct Proof?

When Kant addresses the possibility of mathematical knowledge, he uses both notions of direct proof. Mathematical knowledge for Kant is the knowledge

Dummett, *Elements of Intuitionism*, Clarendon Press, Oxford, 1977. For Kant's view on intuition, see L. Falkeinstein's *Kant's Intuitionism*, University of Toronto Press, 1995. This book, however, does not really address issues related to intuitionism in philosophy of mathematics. I thank Friedman for calling my attention to this book.

[23] This example is taken from M. Kac and S. M. Ulam's *Mathematics and Logic*, (Praeger, New York, 1968). According to Curd, however, the second example from geometry illustrates the notion of existence proof better than the first example. I leave it to the reader to decide the case.

gained by reason from the *construction* of concepts (*Critique*, A 713/B 741). What is it to construct a concept? Kant writes, "[to] construct a concept means to exhibit *a priori* the intuition which corresponds to the concept." For Kant, an intuition is the direct apprehension of an individual object. To provide an existence proof for an object, it is necessary to be able to specify that object. According to Kant, this object must be specified by ostension that is also known as intuition.

Ostension or intuition plays a crucial role in Kant's philosophy. He writes, (*Critique*, B 154) that the possibility of construction insures the "existence" of the object in question. It is important to note that both geometry and arithmetic are "ostensive[24]". He discusses at the passage (B 15-16) how numbers can be ostended by fingers, strokes on a page, and how all of them are spatial representations. In constructing a geometrical figure, a triangle for example, we often represent it by a figure drawn on a black board. In the same way we "construct" arithmetical or algebraic concepts when we represent the individual quantities, perhaps by the fingers of a hand, perhaps by numerals or letters.

It is unclear whether Kant is aware of these distinct senses of direct proofs. When he writes, "[to] construct a concept means to exhibit *a priori* the intuition which corresponds to the concept", he may mean by "constructing a concept", either "an existence proof" or "a constructive proof". Although it is unclear whether Kant knows the distinction, it is evident historically that he is familiar with both indirect and direct kinds of proofs used in mathematics. Further, it is evident from his writings that due to his "transcendental method" he is more favorably disposed to the use of direct proofs rather than indirect proofs after the publication of the *First Critique* (1781). He takes reductio proof as "a last resort" (*Critique*, B 818).[25] Here, objects to be proved fail to be ostended, and therefore they are not possible objects of intuition. It was no wonder that the argument Kant employed in the *Prolegomena* (1783 § 13) was an example of a direct proof. To make his transcendental philosophy consistent with his earlier works, he recycled incongruent-counterparts in the critical period of his career to argue that space is both absolute and intuitive.

So far what I have argued in the last three sections can be summarized as follows: (i) Kant's notions of proof undergo a change over a period of fifteen years. We are able to read this change in the *First Critique* when we come across his unfavorable observations toward indirect proofs as opposed to direct proofs. (ii) There are two senses of a direct proof, (a) an existence proof, and (b) a constructive proof. Finally and (iii), although it is unclear whether Kant is aware of this distinction, it is quite clear that he is favorably disposed to the use of direct proofs.

[24] See, W. Harper, "Kant, Riemann, and Reichenbach on Space and Geometry" in the *Proceedings of the Eighth International Kant Congress*, Vol. 1 part 2, (ed.) H. Robinson, Marquette University Press, 1995 for Kant's views on ostensive constructions and space. I thank Friedman for this reference.

[25] When I mentioned this passage to Friedman, he told me that it seemed like Kant considered a reductio proof as "an invalid argument." (In a private conversion, APA, Pacific Divisional Meeting, 1996). I don't think that for Kant a reductio proof is an analogue of an invalid argument. I have given a justification for Kant's thought in next section.

Why my Interpretation is Better

In the literature on scientific explanation,[26] an explanation is considered to be a good scientific explanation if it is able to unify diverse phenomena. A theory that has the ability to provide a unifying explanation for several apparently unrelated phenomena is hailed as a good theory. For example, we prefer Einstein's theory of relativity to Newton's theory because of this reason. Using Einstein's theory of relativity, we can explain (i) the occurrence of a red-shift, (ii) the bending of light in front of massive nearby objects and finally (iii) the precession rate of Mercury's perihelion with sufficiently precise details. Although the ability to unify diverse items under one banner is canvassed as a plus point for scientific explanation, I think that this ability should also be counted as an added advantage for philosophical accounts that provide a unifying view of several apparently disjoint subjects. I claim that my interpretation has the ability to unify different parts of Kant's views on mathematics. There are two independent considerations for my claim.

(i) Kant takes geometry to be an intuitive enterprise. By this he means that a geometric proof depends on constructing a figure in intuition. In geometry, on the other hand, to show that a proof is impossible, we could often assume to the contrary that the alleged proof is possible. Thus, we end up deriving an inconsistent result from this assumption. An indirect proof depends on constructing a figure in intuition, which turns out not to be a possible object of intuition. How could we resolve this *puzzle* that arises in Kant's transcendental philosophy? One solution to this puzzle is to reject an indirect proof as a valid proof in Kant's system, because we cannot construct a figure that is not a possible object of intuition. This makes Kant's philosophy of mathematics coherent by making his pre-critical writings consistent with the *Prolegomena*.

(ii) It is a well-known metalogical result that if one accepts the law of bivalence, then one is committed to accepting an indirect proof as a valid method of proof. According to the law of bivalence, every proposition must be either true or false. I argued that Kant is skeptical about reductio proofs. For an object to be a possible object of intuition it must be ostended. He argues that for an object to be ostended, it must satisfy two requirements; the

[26] See W. Salmon's article "Scientific Explanation" in W. Salmon et. al. (eds.) *An Introduction to the Philosophy of Science*, Prentice Hall, N.J., 1992. M. Friedman and P. Kitcher have advocated this view about explanation. See, Friedman, "Explanation and Scientific Understanding" in *Journal of Philosophy*, 71: pp. 5-19, 1974. See also, Kitcher, "Explanatory Unification" in *Philosophy of Science*, 48: pp. 507-531, 1981.

requirement of logical possibility and that of real possibility.[27] It is logically possible to imagine that two straight lines can enclose a space. Our imagination satisfies the first requirement, since we do not commit a self-contradiction if we try to imagine so. The former fails to be ostended, because it does not satisfy the second requirement. In a reductio proof, however, neither of the requirements is satisfied. The object to be proved does not get ostended. Therefore, it cannot be a possible object of intuition. This could be a reason why Kant recommends that a reductio proof should be a last resort. It further follows from this that the law of bivalence must be called into question if we use *modus tollens* with the above metalogical result. This, in turn, supports Brittan's contention that Kant dispenses with the law of bivalence.[28]

Summing Up

In the pre-critical period especially when Kant wrote the "Regions in Space", and in the critical period, especially when he wrote the *Prolegomena*, Kant used incongruent-counterparts to argue both that space is absolute and intuitive. I asked a question, "why did he recycle incongruent counterparts to establish the same points about space twice?" Commentators like Meerbote argued that Kant's argument in the *Prolegomena* was intended to establish that space is both relational and intuitive. I argued that this relational reading did injustice to Kant's anti-reductionism in philosophy of mathematics. I argued, to the contrary, that Kant provided in 1768 an indirect proof for absolute theory and he provided in 1783 a direct proof for absolute theory. This change in his concept of proof was due to the emergence of the transcendental philosophy in which his dissatisfaction with indirect proofs was evident. Why does Kant not use incongruent counterparts in the *First Critique* (1781)? Here is the answer. The most relevant aspect of my interpretation rests on exploiting a section of the *Critique* called "Transcendental Doctrine of Method". In this section, Kant commented on both philosophical and mathematical methods. This was written in 1781 and remained unchanged when the *Second Critique* (1787) was published. After the publication of the *First Critique*, Kant's immediate major work was the *Prolegomena* in which Kant first got an opportunity to address incongruent counterparts. And Kant took that opportunity.

I conclude that my account both demystifies Kant's reasons for recycling incongruent counterparts in the *Prolegomena* and provides a unified approach to his philosophy of geometry. Interestingly, my account further defends Beck's interpretation that Kant uses incongruent counterparts in the

[27] Friedman, *Kant and the Exact Sciences*, Harvard University Press, Mass, 1992, pp. 99-101.

[28] See Brittan (this year Kant Congress), "Transcendental Idealism, Empirical Realism, and the Completeness Principle". See also his *Kant's Theory of Science*.

Prolegomena to argue that space is both absolute and intuitive with a rationale for Kant's use of incongruent counterparts, that is missing in Beck's writings.[29]

[29] The author wishes to thank the late L. Beck, J. Allard, G. Brittan Jr., M. Curd, L. Falkeinstein, M. Friedman, J. Earman, R. Meerbote, E. Watkins and P. Weirich for both suggestion and encouragement. The author is especially indebted to Professor Beck for allowing him to present a version of this paper in Beck's summer seminar at the University of Rochester in 1996. He is also thankful to William Donaldson, Michael Mathias, and Phil Mouch for helpful comments on an earlier version of the paper.